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Exact results for the directed Abelian sandpile models

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Abstract. We define the directed Abelian sandpile models by introducing a parameter, c, representing the degree of anisotropy in the avalanche processes, where $c = 1$ is for the isotropic case. We calculate some quantities characterizing the self-organized critical states on the one- and two-dimensional lattices. In particular, we obtain the expected number of topplings per added particle, $\langle T \rangle$, which shows the dependence on the lattice size L as L^x for large L. We show that the critical exponent x does not depend on the dimensionality d, at least for $d = 1$ and 2, and that when any anisotropy is included in the system $x = 1$, while $x = 2$ in the isotropic system. This result gives a rigorous proof of the conjecture by Kadanoff *et al* (1989 *Phys. Rev.* A **39** 6524–37) that the anisotropy will distinguish different universality classes. We introduce a new critical exponent, θ, defined by $\chi \equiv \lim_{L\to\infty} \langle T \rangle / L$ with $c \neq 1$ as $\chi \sim |c-1|^{-\theta}$ for $|c-1| \ll 1$. Both in $d = 1$ and 2, we obtain $\theta = 1$.

1. Introduction and main results

Bak, Tang and Wiesenfeld (BTW) introduced a simple cellular automaton model, whose time-evolution rules capture some aspects of the dynamics of sand grains tumbling on the slope of a sand pile [1, 2]. This model exhibits an extremely attractive property such that without tuning of parameters it produces a unique critical state, which is characterized by power-law correlations and the lack of characteristic sizes of avalanches. The critical state is called the *self-organized critical (SOC) state* [1–3]. The purpose of BTW was not to make models describing the dynamics of real sandpiles, but to study the SOC state. They studied the BTW model using Monte Carlo methods [1, 2] and by the mean-field theory [4, 5].

The BTW paper led experimentalists to study avalanches of real sandpiles. Those experiments revealed that small avalanches have power-law distributions, but behaviours of large avalanches are far from those of critical dynamics [3, 6]. A successful experiment in observing SOC behaviour was performed by Frette *et al* [7]. They studied avalanches of three types of rice instead of sand grains and found that the avalanches of elongated grains show SOC behaviour. It was also reported that the superconduction avalanches and droplet formation show the behaviours having the SOC property [3]. Beyond experiments in laboratories, some natural phenomena such as the distribution of earthquakes and the biological evolution of species are expected to exhibit SOC behaviour [3, 6].

Recently, the object of many theorists has been the Abelian sandpile model (ASM) named by Dhar [8] after its nature that operators representing the processes of adding a particle at a randomly chosen site and relaxing the system form an Abelian group. This feature makes the ASM tractable to analyse. Since the time evolution of the ASM is Markovian, there are two classes of configurations in the SOC state, *recurrent* and *transient*, and only

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recurrent configurations are allowed to occur with probability which is the same for all allowed configurations. Dhar formulated, in terms of the matrix Δ describing toppling rules, the total number of allowed configurations, the entropy of the SOC state and the correlation function of topplings in the SOC state [8].

The isotropic ASM has been well studied. The equivalence between the undirected ASM and the $q \to 0$ limit of the q-state Potts model is established by Majumdar and Dhar [9] by showing the equivalence between the steady state of the ASM and the spanning trees problem which has well known relations to the Potts model. Priezzhev obtained the fractional number of sites having a given height for the two-dimensional ASM [10] by the method developed on this equivalence. The exponents characterizing avalanches have been derived [11] using a more direct representation—'waves of toppling' [12]. Mean-field-type theory was also developed for the ASM and the SOC state in high dimensions were well studied [13–15].

We are interested in the effect of anisotropy on the SOC. Because there are many cases in which the anisotropy of the evolution rules changes the critical exponents; a typical example is the directed percolation transition. For the completely directed case of the ASM, Dhar and Ramaswamy obtained some critical exponents which are defined especially for the systems which extend to infinity in one direction [16]. Kadanoff *et al* performed the computer simulations in both the isotropic and the completely directed cases on the square lattice [17]. Their numerical results show that the expected number of topplings per added particle in the SOC state, $\langle T \rangle$, depends on the system size L as L^x for large L. Based on their simulation results and by the simple analogy of the biased and unbiased random walk problems, Kadanoff *et al* conjectured that the critical exponent $x = 1$, if any anisotropy is included in the system [17], although, in the isotropic case, Dhar obtained $x = 2$ exactly [8]. Kamakura *et al* studied this problem by systematically changing the degree of anisotropy in their computer simulations and reported that their numerical results suggested that the exponent x changes from 2 to 1 when the system has *any* anisotropy [18]. Recently, Head and Rodgers studied the anisotropic Bak–Sneppen model in one dimension and showed that, even when the slightest anisotropy is introduced, the system falls into a different universality class [19].

In this paper, we define the *directed Abelian sandpile model* (DASM) by introducing a parameter representing the degree of anisotropy, c, where the $c = 1$ case corresponds to the isotropic case. We obtain the explicit expressions for the number of the allowed configurations, the entropy and $\langle T \rangle$ in the cases of one and two dimensions using Dhar's formulae. In particular, our results for $\langle T \rangle$ show that

$$x = \begin{cases} 2 & \text{if} \quad c = 1 \\ 1 & \text{if} \quad c \neq 1 \end{cases} \tag{1.1}$$

both in one and two dimensions. This is the proof of the conjecture by Kadanoff *et al*. Moreover, these results allow us to introduce a critical exponent, θ, characterizing the diverging behaviour of the coefficients of $\langle T \rangle$ for the $c \neq 1$ infinite systems, $\chi = \lim_{L\to\infty} \langle T \rangle / L$, as $c \to 1$. We define this exponent as

$$\chi \sim |c - 1|^{-\theta} \qquad \text{for} \quad |c - 1| \ll 1 \tag{1.2}$$

and refer to it as the *anisotropy exponent*. Our exact solutions conclude that $\theta = 1$ both in one and two dimensions. Though we only report the results for the DASM with *nearest-neighbour interactions* in this paper, it is implied that the anisotropy is relevant for the SOC in general and an arbitrarily small amount of anisotropy also changes the exponents in the DASMs with more complicated interactions [19, 20].

Since $\langle T \rangle$ expresses the mean volume of an avalanche, our result (1.1) implies that the avalanches in the directed cases are one-dimensional, while they are two-dimensional in the undirected cases, independent of the spatial dimensionality in which the model is defined. If (1.1) is true for any dimension, it may be concluded that the upper critical dimension d_u is 2 in the $c \neq 1$ cases, although $d_u = 4$ in the $c = 1$ case.

This paper is organized as follows. In section 2 we introduce the DASM on the d-dimensional hypercubic lattices. In section 3.1 we calculate the number of allowed configurations, the entropy and $\langle T \rangle$ of the DASM on a one-dimensional lattice exactly, and section 3.2 is devoted to the two-dimensional case. Some details of the calculations of $\langle T \rangle$ are given in the appendices. Concluding remarks are given in section 4.

2. Definition of the DASM

In this section we give a precise definition of our DASM.

Consider a finite set of sites on a d-dimensional hypercubic lattice with linear size L, $\Lambda_d(L)$, defined by

$$\Lambda_d(L) = \{x = (x_1, \dots, x_d) : x \in \mathbb{Z}^d, 1 \leqslant x_k \leqslant L, k = 1, \dots, d\}. \tag{2.1}$$

To each site $x \in \Lambda_d(L)$ a positive integer variable $z(x)$ is assigned, which can be regarded as the number of grains of sand or the height of the sandpile at site x. The stability of the system is characterized by a set of L^d critical values, $z_c(x)$ for $x \in \Lambda_d(L)$. When $z(x) \leqslant z_c(x)$ is satisfied at all sites, this system is *stable*. When $z(x) > z_c(x)$, site x is said to be over-critical, and the system including such over-critical sites is *unstable*.

Assume that the system starts at an arbitrary stable configuration. The time evolution of this model is the same as that of the ASM, and consists of the following two processes.

(i) Adding a particle.
Add a particle at a randomly selected site $x \in \Lambda_d(L)$. This means that,

$$z(x) \to z(x) + 1 \tag{2.2}$$

and other $z(y)$'s ($y \neq x$) are unchanged. In this procedure, the probability of selecting each site is not necessarily equal. For simplicity, however, we assume that each site is selected with equal probability for adding a particle from now on.

If this system still remains stable, repeat the above procedure until the system becomes unstable. If the system becomes unstable, then every over-critical site *topples* according to the following toppling rules.

(ii) The toppling rules.
The toppling rules are specified in terms of an $L^d \times L^d$ integer matrix Δ. If $z(x) > z_c(x)$, then

$$z(y) \to z(y) - \Delta(x, y) \qquad \text{for all} \quad y \in \Lambda_d(L) \tag{2.3}$$

where $\Delta(x, y)$ is the (x, y)-element of Δ. In general, the diagonal elements of Δ must be positive, and the off-diagonal elements must not be positive. To relax this system to a stable state, some particles may leave $\Lambda_d(L)$, then Δ must satisfy the condition, $\sum_{y \in \Lambda_d(L)} \Delta(x, y) \geqslant 0$ [8].

We impose the open boundary condition so that some particles can leave the system if topplings occur at a corner or edge of the lattice. More precisely, this condition can be written by introducing the sink sites as $\Lambda_d(L+1) \setminus \Lambda_d(L)$, on these sinks no particles are added and no topplings occur. The open boundary condition corresponds to fixing $z(x) \equiv 1\ \forall x \in \Lambda_d(L+1) \setminus \Lambda_d(L)$. This dissipation plays a key role to evolve the system to a steady state.

The topplings continue until the system settles down in a stable configuration, and this series of the topplings are called *an avalanche*. After an avalanche, a new particle is again added to the system.

The anisotropy of this model is introduced by making a preferable direction of dropping. In the DASM, when a toppling occurs at x, then particles on x drop onto its nearest-neighbour sites, $\mathcal{N}_d(x) = \{x \pm e_k : x \in \Lambda_d(L), k = 1, 2, \ldots, d\}$, where $\{e_k\}$ are the bases of the $\mathbb{Z}^d$. Let us divide this site set into two subsets, *positive* and *negative* nearest-neighbour sites, $\mathcal{N}_d^+(x)$ and $\mathcal{N}_d^-(x)$ respectively, where

$$\begin{aligned} \mathcal{N}_d^+(x) &= \{x + e_k : x \in \Lambda_d(L), k = 1, 2, \ldots, d\} \\ \mathcal{N}_d^-(x) &= \{x - e_k : x \in \Lambda_d(L), k = 1, 2, \ldots, d\}. \end{aligned} \tag{2.4}$$

Let c be a positive rational number and ζ be a positive integer such that $c\zeta$ is a positive integer. We assume that $z_c(x) = d(c+1)\zeta$ for all $x \in \Lambda_d(L)$, and in a toppling at x, $c\zeta$ particles drop onto $\mathcal{N}_d^+(x)$ while ζ particles drop onto $\mathcal{N}_d^-(x)$. Thus the (x, y)-element of Δ can be written as follows for all $x, y \in \Lambda_d(L)$.

$$\Delta(x, y) = \begin{cases} d(c+1)\zeta & \text{if} \quad y = x \\ -c\zeta & \text{if} \quad y \in \mathcal{N}_d^+(x) \\ -\zeta & \text{if} \quad y \in \mathcal{N}_d^-(x) \\ 0 & \text{otherwise.} \end{cases} \tag{2.5}$$

Note that c represents the degree of anisotropy, and the $c = 1$ case corresponds to the isotropic case. By choosing the ζ appropriately we can consider the weakly anisotropic case. For example, for $c = 1.1$, we may take $\zeta = 10$, and then at a toppling, $c\zeta = 11$ particles drop onto the positive nearest-neighbour sites and $\zeta = 10$ particles in the negative nearest-neighbour sites.

It can be easily confirmed that the (c, ζ)-DASM with $c > 1$ is the same as the $(1/c, c\zeta)$-DASM. Therefore, without loss of generality, we can assume $c > 1$.

3. Exact calculations

In this section we calculate several quantities characterizing the SOC state of the DASM. For the ASM, the number of allowed configurations, N_R, is generally given by [8, 9, 21]

$$N_R = \det \Delta. \tag{3.1}$$

Dhar defined the entropy, S, and the entropy per site in the infinite-volume limit, s, as

$$S = \log N_R \tag{3.2}$$

$$s = \lim_{L\to\infty} \frac{S}{L^d}. \tag{3.3}$$

Let $G(x, y)$ be the average number of the topplings at y caused due to adding a particle at site x, then [8]

$$G(x, y) = [\Delta]^{-1}(x, y) \tag{3.4}$$

where Δ^{-1} is the inverse matrix of Δ. Thus the expected number of topplings per added particle, $\langle T\rangle$, can be written as

$$\langle T\rangle = \frac{1}{L^d} \sum_{x\in\Lambda_d(L)} \sum_{y\in\Lambda_d(L)} G(x, y). \tag{3.5}$$

3.1. One-dimensional DASM

To calculate eigenvalues of Δ and Δ^{-1}, we first diagonalize Δ. Since Δ is not symmetric, the diagonalized matrix, Λ, is obtained using two matrices P and Q, $\Lambda = P\Delta Q$. Thus, if $PQ = QP = \mathrm{E}$, then, $\Delta^{-1} = Q\Lambda^{-1}P$. Let $P(n, x)$ and $Q(x, n)$ be the elements of P and Q, respectively. We found that for $x, n = 1, 2, \ldots, L$,

$$P(n, x) = \sqrt{\frac{2}{L+1}} c^{\frac{x}{2}} \sin\left(\frac{nx}{L+1}\pi\right) \tag{3.6}$$

$$Q(x, n) = \sqrt{\frac{2}{L+1}} c^{-\frac{x}{2}} \sin\left(\frac{nx}{L+1}\pi\right). \tag{3.7}$$

It follows that the (n, n')-element of Λ,

$$\Lambda(n, n') = \delta_{n,n'}\lambda(n) \tag{3.8}$$

with

$$\lambda(n) = \left\{c + 1 - 2\sqrt{c}\cos\left(\frac{n}{L+1}\pi\right)\right\}\zeta. \tag{3.9}$$

From (3.1)

$$\begin{aligned} N_R &= \prod_{n=1}^{L} \lambda(n) \\ &= \frac{c^{L+1} - 1}{c - 1}\zeta^L. \end{aligned} \tag{3.10}$$

Here we have used the formula,

$$\prod_{r=1}^{n-1}\left(x^2 - 2x\cos\left(\frac{r\pi}{n}\right) + 1\right) = \frac{x^{2n} - 1}{x^2 - 1}. \tag{3.11}$$

In order to take the isotropic limit $c \to 1$ we first expand (3.10) with respect to c about $c = 1$,

$$N_R = (L+1)\zeta^L[1 + \tfrac{1}{2}L(c-1) + \mathrm{O}((c-1)^2)] \tag{3.12}$$

and then we have

$$\begin{aligned} N_{R1} &\equiv \lim_{c\to 1} N_R \\ &= (L+1)\zeta^L. \end{aligned} \tag{3.13}$$

From (3.2) and (3.3), we have for general $c > 1$

$$S = L\ln(\zeta) + \ln\left(\frac{c^{L+1} - 1}{c - 1}\right) \tag{3.14}$$

$$s = \ln c\zeta. \tag{3.15}$$

Using (3.6), (3.7) and (3.9) Δ^{-1} can be obtained and substituting it into (3.5) gives $\langle T\rangle$ as

$$\begin{aligned} \langle T\rangle &= \frac{2}{L(L+1)\zeta}\sum_{n=1}^{L}\frac{1}{c+1-2\sqrt{c}\cos\left(\frac{n}{L+1}\pi\right)}\sum_{x_1=1}^{L} c^{-\frac{x_1}{2}}\sin\left(\frac{nx_1}{L+1}\pi\right)\sum_{x_2=1}^{L} c^{\frac{x_2}{2}}\sin\left(\frac{nx_2}{L+1}\pi\right) \\ &= \frac{2c}{L(L+1)\zeta}\sum_{n=1}^{L}\{2 - (-1)^n(c^{\frac{L+1}{2}} + c^{-\frac{L+1}{2}})\}\sin^2\left(\frac{n}{L+1}\pi\right) \\ &\quad\times\left[\frac{1}{1 + c - 2\sqrt{c}\cos\left(\frac{n}{L+1}\pi\right)}\right]^3. \end{aligned} \tag{3.16}$$

From this expression, $\langle T\rangle$ seems to diverge exponentially as $c^{\frac{L}{2}}$ when $L \gg 1$. However, $L^3 c^{-\frac{L}{2}}$ comes out by performing the summation, and such exponential factors will be completely cancelled. It is remarkable that the approximation of the summation with an integral fails to derive the $L^3 c^{-\frac{L}{2}}$ terms. The way to perform the summation is given in appendix A, and the final expression is

$$\langle T\rangle = \frac{1}{2\zeta L}\left[\frac{1}{c-1} \times \frac{1+c^{-(L+1)}}{1-c^{-(L+1)}}(L+1)^2 - \frac{c+1}{(c-1)^2}(L+1)\right] \tag{3.17}$$

$$\simeq \frac{L}{2\zeta(c-1)} \qquad \text{for} \quad L \gg 1. \tag{3.18}$$

Thus we obtain $x = 1$ and $\theta = 1$.

In order to take the isotropic limit $c \to 1$ we first expand (3.17) with respect to c about $c = 1$,

$$\langle T\rangle = \frac{1}{2L\zeta}\left[\frac{L(L+1)(L+2)}{6} - \frac{(L+1)(L+2)}{12}(c-1) + \mathrm{O}((c-1)^2)\right]. \tag{3.19}$$

Although (3.17) seems to be singular at $c = 1$, this expression has no singularity at $c = 1$, since the singular part of each term in (3.17) cancel each other. Thus we obtain,

$$\langle T\rangle_1 \equiv \lim_{c\to 1}\langle T\rangle$$

$$= \frac{L(L+1)(L+2)}{12L\zeta}$$

$$\simeq \frac{L^2}{12\zeta} \qquad \text{for} \quad L \gg 1. \tag{3.20}$$

This means that $x = 2$ for $c = 1$.

It is remarkable that those expressions can be obtained from the (3.17) by taking $c \to 1$ and $L \to \infty$ only in this turn, since the $L \gg 1$ approximation eliminates the partner in the offsetting of singular parts in the expansion about $c = 1$.

3.2. Two-dimensional DASM

We choose the elements of P and Q matrices as follows:

$$P(n, x) = \frac{2}{L+1} c^{\frac{x_1+x_2}{2}} \sin\left(\frac{n_1 x_1}{L+1}\pi\right) \sin\left(\frac{n_2 x_2}{L+1}\pi\right) \tag{3.21}$$

$$Q(x, n) = \frac{2}{L+1} c^{-\frac{x_1+x_2}{2}} \sin\left(\frac{n_1 x_1}{L+1}\pi\right) \sin\left(\frac{n_2 x_2}{L+1}\pi\right) \tag{3.22}$$

where $x = (x_1, x_2)$ and $n = (n_1, n_2)$. From these expressions, the eigenvalues of Δ can be obtained as

$$\lambda(n) = 2\zeta\left\{c + 1 - \sqrt{c}\left(\cos\left(\frac{n_1}{L+1}\pi\right) + \cos\left(\frac{n_2}{L+1}\pi\right)\right)\right\}. \tag{3.23}$$

From the formula (3.3), the entropy per site in the infinite-volume limit, s, is given as

$$s = \ln 2\zeta + \frac{1}{\pi^2}\int_0^{\pi}\int_0^{\pi} \mathrm{d}\theta\, \mathrm{d}\phi\ \ln\left[c + 1 - \sqrt{c}(\cos\theta + \cos\phi)\right]. \tag{3.24}$$

The inverse matrix of Δ can be calculated with the above expressions for P, Q and $\lambda(n)$, and using the formula (3.5) gives

$$\langle T\rangle = \frac{2c^2}{L^2(L+1)^2\zeta}\sum_{n_1=1}^{L}\sum_{n_2=1}^{L}\frac{\sin^2\left(\frac{n_1\pi}{L+1}\right)\sin^2\left(\frac{n_2\pi}{L+1}\right)}{c + 1 - \sqrt{c}\left(\cos\left(\frac{n_1\pi}{L+1}\right) + \cos\left(\frac{n_2\pi}{L+1}\right)\right)}$$

$$\times \frac{2 - (-1)^{n_1}\left(c^{\frac{L+1}{2}} + c^{-\frac{L+1}{2}}\right)}{\left\{1 + c - 2\sqrt{c}\cos\left(\frac{n_1\pi}{L+1}\right)\right\}^2} \times \frac{2 - (-1)^{n_2}\left(c^{\frac{L+1}{2}} + c^{-\frac{L+1}{2}}\right)}{\left\{1 + c - 2\sqrt{c}\cos\left(\frac{n_2\pi}{L+1}\right)\right\}^2}. \tag{3.25}$$

By performing the double summations carefully, we obtain

$$\langle T\rangle \simeq \frac{1}{3(c-1)\zeta}L \qquad \text{for} \quad L \gg 1. \tag{3.26}$$

The details of the derivation are explained in appendix B.

From this expression, we can conclude that

$$x = 1 \qquad \text{for} \quad c \neq 1 \quad \text{and} \quad \theta = 1. \tag{3.27}$$

It should be remarked that the result (3.24) shows that the entropy of the SOC is enhanced by the anisotropy but the dimensionality of the avalanches in the SOC is reduced from $x = 2$ to $x = 1$.

4. Concluding remarks

The explicit expression (3.17) for the one-dimensional $\langle T\rangle$ is very useful to see how the critical exponent x changes from 1 to 2 as c approaches 1. Let

$$f(c, L) = \langle T\rangle \times \frac{2\zeta(c-1)}{L}. \tag{4.1}$$

We define the *scaling limit*, $\lim_{\text{scaling}}$, as the double limits $c \to 1$ and $L \to \infty$ keeping $(c-1)L = z$ a constant. From (3.17), we obtain

$$f_{\text{scaling}}(z) \equiv \lim_{\text{scaling}} f(c, L) \tag{4.2}$$

$$= \coth\left(\frac{z}{2}\right) - \frac{2}{z}. \tag{4.3}$$

The scaling limit of $\langle T\rangle$ is now defined as

$$\langle T\rangle_{\text{scaling}} = \frac{L}{2\zeta(c-1)} f_{\text{scaling}}(z). \tag{4.4}$$

Note that the $z \to \infty$ limit means taking the $L \to \infty$ limit keeping $c \neq 1$, which corresponds to the anisotropic case. On the other hand, the $z \to 0$ limit means taking the $c \to 1$ limit faster than the infinite-size limit, which results in the isotropic case. From (4.3) and (4.4), it is immediately given that

$$\lim_{z\to\infty} \langle T\rangle_{\text{scaling}} = \frac{L}{2\zeta(c-1)} \tag{4.5}$$

$$\lim_{z\to 0} \langle T\rangle_{\text{scaling}} = \frac{L^2}{12\zeta}. \tag{4.6}$$

They correspond to the expressions (3.18) and (3.20).

Recently, Shimamura *et al* studied the DASM by computer simulations [22]. The numerical data support our results, $\langle T\rangle \sim L^x$ with (1.1) both in $d = 1$ and 2. Moreover, Shimamura *et al* observed that in the $d = 2$ case the distribution function of the number of topplings per added particle, P_T, also obeys the power-law $T^{-\tau}$ for the directed cases $c \neq 1$, but that in the $d = 1$ and $c \neq 1$ case, P_T seems to be the white noise, i.e. $P_T \sim$ const., and in the case of $d = 1$ and $c = 1$ the data of P_T scatter even in the large systems. It is interesting that, although the average values, $\langle T\rangle$, show the same critical behaviours for large L and $|c-1| \ll 1$ both in $d = 1$ and 2 cases, the distribution, P_T, exhibits such variety and sensibility depending on the dimensionality and the anisotropy. More details will be published in the forthcoming paper [22].

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Appendix A. Derivation of $\langle T\rangle$ in the one-dimensional case

First we define the quantity

$$R_\rho^{(i)}(x,m) = \frac{\sin^2 k_\rho(m)\cos^2 k_\rho(m)}{(x+\sin^2 k_\rho(m))^i} \tag{A.1}$$

for an arbitrary integer i and $\rho = 1, 2$, where

$$k_1(m) = \frac{2m-1}{2(L+1)}\pi \qquad \text{and} \qquad k_2(m) = \frac{m}{L+1}\pi \tag{A.2}$$

and consider the summation

$$S_\rho^{(i)}(x) = \sum_{m=1}^{[\frac{L+1}{2}]} R_\rho^{(i)}(x,m) \tag{A.3}$$

for $\rho = 1, 2$. Then we divide the summation of (3.16) into the cases of $n =$ even and odd as

$$\langle T\rangle = \frac{c^{-\frac{1}{2}}}{8L(L+1)\zeta}[\{2+(c^{\frac{L+1}{2}}+c^{-\frac{L+1}{2}})\}S_1^{(3)}(\beta^2) + \{2-(c^{\frac{L+1}{2}}+c^{-\frac{L+1}{2}})\}S_2^{(3)}(\beta^2)] \tag{A.4}$$

where

$$\beta = \frac{c^{\frac{1}{4}} - c^{-\frac{1}{4}}}{2}. \tag{A.5}$$

Thus we must obtain $S_1^{(3)}(\beta^2)$ and $S_2^{(3)}(\beta^2)$.

We can confirm the following identities for any integer $n \geqslant 1$,

$$\begin{aligned}\tanh(2n+1)x &= \frac{\tanh x}{2n+1} + \frac{2}{2n+1}\sum_{r=0}^{n-1}\frac{\tanh x}{\sin^2\left(\frac{(2r+1)\pi}{4n+2}\right) + \cos^2\left(\frac{(2r+1)\pi}{4n+2}\right)\tanh^2 x}\\ \coth 2nx &= \frac{1}{n}\coth 2x + \frac{1}{n}\sum_{r=1}^{n-1}\frac{\tanh x}{\sin^2\left(\frac{r\pi}{2n}\right)+\cos^2\left(\frac{r\pi}{2n}\right)\tanh^2 x}.\end{aligned} \tag{A.6}$$

Using these identities we obtain

$$\begin{aligned}S_1^{(1)}(x) &= \frac{L+1}{4}g(x)^{-\frac{1}{2}}\frac{1+g(x)^{-\frac{1}{2}(L-1)}}{1+g(x)^{-\frac{1}{2}(L+1)}}\\ S_2^{(1)}(x) &= \frac{L+1}{4}g(x)^{-\frac{1}{2}}\frac{1-g(x)^{-\frac{1}{2}(L-1)}}{1-g(x)^{-\frac{1}{2}(L+1)}}\end{aligned} \tag{A.7}$$

where

$$g(x) = \left(\sqrt{x}+\sqrt{x+1}\right)^4. \tag{A.8}$$

Since $g(\beta^2) = c$, we have

$$\begin{aligned}S_1^{(1)}(\beta^2) &= \frac{L+1}{4}c^{-\frac{1}{2}}\frac{1+c^{-\frac{1}{2}(L-1)}}{1+c^{-\frac{1}{2}(L+1)}}\\ S_2^{(1)}(\beta^2) &= \frac{L+1}{4}c^{-\frac{1}{2}}\frac{1-c^{-\frac{1}{2}(L-1)}}{1-c^{-\frac{1}{2}(L+1)}}.\end{aligned} \tag{A.9}$$

In general, $S_1^{(i)}$, $S_2^{(i)}$ for $i \geqslant 1$ can be calculated by determining their generating functions. Define $\hat{S}_\rho^{(i)}(a)$ $(\rho = 1, 2)$ as

$$\begin{aligned}\hat{S}_\rho^{(i)}(a) &= \alpha^{2i} S_\rho^{(i)}(\beta^2)\\ &= \sum_{m=1}^{[\frac{L+1}{2}]} \frac{\sin^2 k_\rho(m)\cos^2 k_\rho(m)}{(\sin k_\rho(m) + a^2 \cos k_\rho(m))^i}\end{aligned} \tag{A.10}$$

where

$$\alpha = \frac{c^{\frac{1}{4}} + c^{-\frac{1}{4}}}{2} \qquad \text{and} \qquad a = \frac{\beta}{\alpha} = \frac{c^{\frac{1}{4}} - c^{-\frac{1}{4}}}{c^{\frac{1}{4}} + c^{-\frac{1}{4}}}. \tag{A.11}$$

The generating functions, $H_\rho(z, a)$, $\rho = 1, 2$, are defined as

$$H_\rho(z, a) = \sum_{i=2}^{\infty} (i-1)\hat{S}_\rho^{(i)}(a) z^{i-2}. \tag{A.12}$$

We can derive partial differential equations for $H_\rho(z, a)$. Using (A.9) as the initial conditions at $z = 0$, we solve them as

$$\begin{aligned}H_1(z,a) &= \frac{L+1}{4(a^2-1)^2}\left[(L+1)\,\mathrm{sech}^2\left((L+1)\,\mathrm{arctanh}\sqrt{\frac{a^2-z}{1-z}}\right)\right.\\ &\quad\left. + \frac{2z-a^2-1}{\sqrt{(z-a^2)(z-1)}}\tanh\left((L+1)\,\mathrm{arctanh}\sqrt{\frac{a^2-z}{1-z}}\right) - 2\right]\\ H_2(z,a) &= -\frac{L+1}{4(a^2-1)^2}\left[(L+1)\,\mathrm{cosech}^2\left((L+1)\,\mathrm{arctanh}\sqrt{\frac{a^2-z}{1-z}}\right)\right.\\ &\quad\left. - \frac{2z-a^2-1}{\sqrt{(z-a^2)(z-1)}}\coth\left((L+1)\,\mathrm{arctanh}\sqrt{\frac{a^2-z}{1-z}}\right) + 2\right].\end{aligned} \tag{A.13}$$

From these generating functions, we can derive $S_\rho^{(k)}$ as

$$S_\rho^{(k+2)}(\beta^2) = \alpha^{-2(k+2)} \frac{1}{(k+1)!}\left[\frac{\partial^k}{\partial z^k} H_\rho(z,a)\right]_{z=0} \tag{A.14}$$

for any $k \geqslant 0$ and $\rho = 1, 2$. In particular we obtain the $i = 3$ case,

$$\begin{aligned}S_1^{(3)}(\beta^2) &= 4c^{\frac{3}{2}}\frac{1-c^{-\frac{1}{2}(L+1)}}{1+c^{-\frac{1}{2}(L+1)}}\frac{L+1}{(c-1)^3} - 2c^{\frac{1}{2}}\frac{(c+1)c^{-\frac{1}{2}(L+1)}}{(1+c^{-\frac{1}{2}(L+1)})^2}\frac{(L+1)^2}{(c-1)^2}\\ &\quad +2c^{\frac{1}{2}}\frac{(1-c^{-\frac{1}{2}(L+1)})c^{-\frac{1}{2}(L+1)}}{(1+c^{-\frac{1}{2}(L+1)})^3}\frac{(L+1)^3}{c-1}\\ S_2^{(3)}(\beta^2) &= 4c^{\frac{3}{2}}\frac{1+c^{-\frac{1}{2}(L+1)}}{1-c^{-\frac{1}{2}(L+1)}}\frac{L+1}{(c-1)^3} + 2c^{\frac{1}{2}}\frac{(c+1)c^{-\frac{1}{2}(L+1)}}{(1-c^{-\frac{1}{2}(L+1)})^2}\frac{(L+1)^2}{(c-1)^2}\\ &\quad -2c^{\frac{1}{2}}\frac{(1+c^{-\frac{1}{2}(L+1)})c^{-\frac{1}{2}(L+1)}}{(1-c^{-\frac{1}{2}(L+1)})^3}\frac{(L+1)^3}{c-1}.\end{aligned} \tag{A.15}$$

Substituting these expressions in (3.16) immediately leads to the final expressions (3.17).

Appendix B. Asymptote of $\langle T \rangle$ for large L in the two-dimensional case

Let

$$J_{\sigma\rho} = \sum_{m_1=1}^{[\frac{L+1}{2}]} \sum_{m_2=1}^{[\frac{L+1}{2}]} \frac{\sin^2 k_\sigma(m_1)\cos^2 k_\sigma(m_1)\sin^2 k_\rho(m_2)\cos^2 k_\rho(m_2)}{\beta^2 + \sin^2 k_\sigma(m_1) + \beta^2 + \sin^2 k_\rho(m_2)} \times \frac{1}{(\beta^2 + \sin^2 k_\sigma(m_1))^2} \times \frac{1}{(\beta^2 + \sin^2 k_\rho(m_2))^2} \tag{B.1}$$

for $\sigma, \rho = 1, 2$. It is clear that $J_{\mu\nu} = J_{\nu\mu}$. Define

$$\begin{aligned} J_1 &= J_{11} - J_{12} - J_{21} + J_{22} \\ J_2 &= J_{11} - J_{22} \\ J_3 &= 3J_{11} + J_{12} + J_{21} + 3J_{22}. \end{aligned} \tag{B.2}$$

Then (3.25) can be written as

$$\langle T \rangle = \frac{c^{-\frac{1}{2}}}{16\zeta L^2(L+1)^2} \times K \tag{B.3}$$

with

$$K = J_1 c^{L+1} + 4J_2 c^{\frac{L+1}{2}} + 2J_3 + 4J_2 c^{-\frac{L+1}{2}} + J_1 c^{-(L+1)}. \tag{B.4}$$

Dividing the fractional expression in $J_{\rho,\sigma}$ given by (B.1) into three partial fractions, we have

$$J_{\sigma\rho} = J_{\sigma\rho}^{(1)} + J_{\sigma\rho}^{(2)} - J_{\sigma\rho}^{(3)} \tag{B.5}$$

where

$$\begin{aligned} J_{\sigma\rho}^{(1)} &= S_\sigma^{(3)}(\beta^2) \times S_\rho^{(2)}(\beta^2) \\ J_{\sigma\rho}^{(2)} &= \sum_{m=1}^{[\frac{L+1}{2}]} R_\sigma^{(4)}(\beta^2, m) S_\rho^{(1)}(2\beta^2 + \sin^2 k_\sigma(m)) \\ J_{\sigma\rho}^{(3)} &= S_\sigma^{(4)}(\beta^2) \times S_\rho^{(1)}(\beta^2). \end{aligned} \tag{B.6}$$

We define for $k = 1, 2, 3$

$$\begin{aligned} J_1^{(k)} &= J_{11}^{(k)} - J_{12}^{(k)} - J_{21}^{(k)} + J_{22}^{(k)} \\ J_2^{(k)} &= J_{11}^{(k)} - J_{22}^{(k)} \\ J_3^{(k)} &= 3J_{11}^{(k)} + J_{12}^{(k)} + J_{21}^{(k)} + 3J_{22}^{(k)}. \end{aligned} \tag{B.7}$$

Using (A.13) and (A.14), we can obtain $S_\rho^{(2)}(\beta^2)$ as well as $S_\rho^{(3)}(\beta^2)$ given by (A.15) and it can be proved that

$$\begin{aligned} &\lim_{L\to\infty} \frac{c^{L+1}}{(L+1)^5} J_1^{(1)} = \frac{8c^{\frac{1}{2}}}{c-1} \\ &\lim_{L\to\infty} \frac{c^{\frac{L+1}{2}}}{(L+1)^5} J_2^{(1)} = 0 \\ &\lim_{L\to\infty} \frac{1}{(L+1)^5} J_3^{(1)} = 0. \end{aligned} \tag{B.8}$$

Let

$$F(x) = \frac{L+1}{2} \frac{\sinh(\frac{1}{2}\ln g(x))}{\sinh(\frac{L+1}{2}\ln g(x))} \tag{B.9}$$

where $g(x)$ is defined by (A.8). Then

$$J_1^{(2)} = \sum_{m=1}^{[\frac{L+1}{2}]} R_1^{(4)}(\beta^2, m) F(2\beta^2 + \sin^2 k_1(m)) - \sum_{m=1}^{[\frac{L+1}{2}]} R_2^{(4)}(\beta^2, m) F(2\beta^2 + \sin^2 k_2(m)). \quad \text{(B.10)}$$

We expand $F(2\beta^2 + \sin^2 k_\rho(m))$ about β^2. For $M \geqslant 4$, Taylor's theorem gives

$$F(2\beta^2 + \sin^2 k_\rho(m)) = F(\beta^2) + \sum_{n=1}^{M-1} \frac{1}{n!} \left[\frac{\mathrm{d}^n F(x)}{\mathrm{d}x^n}\right]_{x=\beta^2} \times (\beta^2 + \sin^2 k_\rho(m))^n$$
$$+ \frac{1}{M!} \left[\frac{\mathrm{d}^M F(x)}{\mathrm{d}x^M}\right]_{x=\beta^2+\theta(\beta^2+\sin^2 k_\rho(m))} \times (\beta^2 + \sin^2 k_\rho(m))^M \quad \text{(B.11)}$$

with $0 <^{\exists} \theta < 1$. Substituting this into (B.10) and the definition (A.3) gives

$$J_1^{(2)} = J_1^{(3)} + \sum_{n=1}^{M-1} \frac{1}{n!} \left[\frac{\mathrm{d}^n F(x)}{\mathrm{d}x^n}\right]_{x=\beta^2} \times (S_1^{(4-n)}(\beta^2) - S_2^{(4-n)}(\beta^2)) + \frac{1}{M!} R_M \quad \text{(B.12)}$$

with

$$R_M = \sum_{m=1}^{[\frac{L+1}{2}]} R_1^{(4-M)}(\beta^2, m) \left[\frac{\mathrm{d}^M F(x)}{\mathrm{d}x^M}\right]_{x=\beta^2+\theta(\beta^2+\sin^2 k_1(m))}$$
$$- \sum_{m=1}^{[\frac{L+1}{2}]} R_2^{(4-M)}(\beta^2, m) \left[\frac{\mathrm{d}^M F(x)}{\mathrm{d}x^M}\right]_{x=\beta^2+\theta(\beta^2+\sin^2 k_2(m))}. \quad \text{(B.13)}$$

It is easy to prove that for any $i \leqslant 0$

$$S_1^{(i)}(\beta^2) - S_2^{(i)}(\beta^2) = 0 \quad \text{(B.14)}$$

if $L \geqslant |i| + 2$. Then we have

$$J_1^{(2)} = J_1^{(3)} + \sum_{n=1}^{3} \frac{1}{n!} \left[\frac{\mathrm{d}^n F(x)}{\mathrm{d}x^n}\right]_{x=\beta^2} \times (S_1^{(4-n)}(\beta^2) - S_2^{(4-n)}(\beta^2)) + \frac{1}{M!} R_M \quad \text{(B.15)}$$

for $L \geqslant M - 1$. Since this holds for arbitrary $M \geqslant 4$, we conclude that

$$J_1^{(2)} - J_1^{(3)} = \sum_{n=1}^{3} \frac{1}{n!} \left[\frac{\mathrm{d}^n F(x)}{\mathrm{d}x^n}\right]_{x=\beta^2} \times (S_1^{(4-n)}(\beta^2) - S_2^{(4-n)}(\beta^2)) \quad \text{(B.16)}$$

for $L \gg 1$. Using (A.7), (A.13), (A.14) and (B.9), we can show that

$$\lim_{L\to\infty} \frac{c^{L+1}}{(L+1)^5} (J_1^{(2)} - J_1^{(3)}) = -\frac{8}{3} \frac{c^{\frac{1}{2}}}{c-1}. \quad \text{(B.17)}$$

Next we let

$$g_m = g(2\beta^2 + \sin^2 k_1(m)). \quad \text{(B.18)}$$

Then (A.7) gives

$$S_1^{(1)}(g_m) = \frac{L+1}{4} \left[g_m^{-\frac{1}{2}} - 2\sinh\left(\frac{1}{2}\ln g_m\right) \times \sum_{n=1}^{\infty} (-1)^n g_m^{-\frac{1}{2}(L+1)n} \right]. \quad \text{(B.19)}$$

Since $g_m > c$ for $c > 1$, we can prove that

$$\lim_{L\to\infty} \frac{c^{\frac{L+1}{2}}}{(L+1)^5} J_{11}^{(2)} = \frac{1}{4} \lim_{L\to\infty} \frac{c^{\frac{L+1}{2}}}{(L+1)^4} \sum_{m=1}^{[\frac{L+1}{2}]} R_1^{(4)}(\beta^2, m) g_m^{-\frac{1}{2}}. \quad \text{(B.20)}$$

In the same way, we obtain the similar equation for $J_{22}^{(2)}$. Following the same argument deriving (B.16), we obtain

$$\lim_{L\to\infty} \frac{c^{\frac{L+1}{2}}}{(L+1)^5}(J_2^{(2)} - J_2^{(3)}) = \frac{1}{4}\lim_{L\to\infty} \frac{c^{\frac{L+1}{2}}}{(L+1)^4}\sum_{n=1}^{3}\left[\frac{\mathrm{d}^n}{\mathrm{d}x^n} g(x)^{-\frac{1}{2}}\right]_{x=\beta^2}$$
$$\times (S_1^{(4-n)}(\beta^2) - S_2^{(4-n)}(\beta^2)). \tag{B.21}$$

In this case, we find that this limit is zero. In the same way, we can also prove that

$$\lim_{L\to\infty} \frac{1}{(L+1)^5}(J_3^{(2)} - J_3^{(3)}) = 0. \tag{B.22}$$

Since

$$J_\alpha = J_\alpha^{(1)} + J_\alpha^{(2)} - J_\alpha^{(3)} \tag{B.23}$$

for $\alpha = 1, 2, 3$. These results give that

$$\begin{aligned} &\lim_{L\to\infty} \frac{c^{L+1}}{(L+1)^5} J_1 = \frac{16}{3}\frac{c^{\frac{1}{2}}}{c-1} \\ &\lim_{L\to\infty} \frac{c^{\frac{L+1}{2}}}{(L+1)^5} J_2 = 0 \\ &\lim_{L\to\infty} \frac{1}{(L+1)^5} J_3 = 0. \end{aligned} \tag{B.24}$$

Then we obtain

$$\lim_{L\to\infty} \frac{1}{(L+1)^5} K = \frac{16}{3}\frac{c^{\frac{1}{2}}}{c-1} \tag{B.25}$$

which gives (3.26) through (B.3).

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